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# GENERALIZED TABLE ALGEBRAS

BY

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ABSTRACT

A table algebra was defined in [1] in order to consider in a uniform way the common properties of conjugacy classes and irreducible characters. Non-commutative table algebras were introduced in [5]. They generalize properties of such well-known objects as coherent and Hecke algebras. Here we extend the main definition of a non-commutative table algebra by letting the ground field be an integral domain. We call these algebras **generalized table** algebras (GT-algebras, in brief). It is worth mentioning that this class of algebras includes generic Hecke–Iwahori algebras of finite Coxeter groups. We develop the general theory for this type of algebras which includes their representation theory and theory of closed subsets. We also study the properties of primitive integral table algebras.

## 1. Introduction

Let $R$ be an integral domain. An $R$-algebra $A$ with a distinguished basis $\mathbf{B}$ is called a **generalized table algebra** (briefly, GT-algebra) if it satisfies the following axioms:

T0. $A$ is a free left $R$-module with a basis $\mathbf{B}$ and $\mathbf{B}$ is finite.

T1. $A$ is an $R$-algebra with unit $\mathbf{1}$, and $\mathbf{1} \in \mathbf{B}$.

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T2. There exists an antiautomorphism $a \to a^*, a \in A$, such that $(a^*)^* = a$ holds for all $a \in A$ and $\mathbf{B}^* = \mathbf{B}$.†

Let $\lambda_{abc} \in R$ be the structure constants of $A$ in basis $\mathbf{B}$, i.e.,

$$ab = \sum_{c \in \mathbf{B}} \lambda_{abc} c, \quad a, b \in \mathbf{B}.$$

T3. For each $a, b \in \mathbf{B}$, $\lambda_{ab\mathbf{1}} = \lambda_{ba\mathbf{1}}$, and $\lambda_{ab\mathbf{1}} = 0$ if $a \neq b^*$.

In what follows, the notation $(A, \mathbf{B})$ will mean a GT-algebra $A$ with the distinguished basis $\mathbf{B}$.

Following [7] we call a basis $\mathbf{B}$ **standard** if the map $b \mapsto \lambda_{bb^*\mathbf{1}}$ extends linearly to a homomorphism of $R$-algebras. We also say that $(A, \mathbf{B})$ is **standard** if $\mathbf{B}$ is standard.

A GT-algebra will be called **real** if $R = \mathbb{R}$ and $\lambda_{abc} \geq 0$ for each triple $a, b, c \in \mathbf{B}$.

Let $t$: $A \to R$ be the linear function defined by $t(\sum_{b \in \mathbf{B}} x_b b) = x_{\mathbf{1}}$. As a direct consequence of T3, we obtain that $t(xy) = t(yx), x, y \in A$.

We shall say that a basis $\mathbf{B}$ (an algebra $A$) is **non-singular** if $\lambda_{bb^*\mathbf{1}} \neq 0$ for each $b \in \mathbf{B}$. In this case $A$ becomes a Frobenius algebra, since $t(xy)$ is a non-degenerate associative form on $A$.

In what follows we use the notation $|x|, x = \sum_{b \in \mathbf{B}} x_b b$ for the sum $\sum_{b \in \mathbf{B}} x_b \lambda_{bb^*\mathbf{1}}$. In particular, $|b| = \lambda_{bb^*\mathbf{1}}$ for each $b \in \mathbf{B}$. If $\mathbf{C} \subset \mathbf{B}$, then $|\mathbf{C}|$ will stand for the sum $\sum_{c \in \mathbf{C}} |c|$.

We define a bilinear form ( , ) on $A$ by setting

$$(x, y) = t(xy^*).$$

According to T3, ( , ) is a symmetric bilinear form, values of which may be computed by the following formula:

$$\left( \sum_{b \in \mathbf{B}} x_b b, \sum_{b \in \mathbf{B}} y_b b \right) = \sum_{b \in \mathbf{B}} x_b y_b \lambda_{bb^*\mathbf{1}}. \tag{1}$$

For any $x = \sum_{b \in \mathbf{B}} x_b b \in A$, we write $\mathrm{Supp}(x)$ for $\{b \in \mathbf{B} \mid x_b \neq 0\}$. If $\mathbf{E}, \mathbf{D} \subset \mathbf{B}$, then we set $\mathbf{ED} = \bigcup_{c \in \mathbf{E}, d \in \mathbf{D}} \mathrm{Supp}(cd)$. We shall write $a\mathbf{C}$ instead of $\{a\}\mathbf{C}, \mathbf{C} \subset \mathbf{B}$. For every $\mathbf{C} \subset \mathbf{B}$ we write $\mathbf{C}^+$ for the following sum, $\sum_{c \in \mathbf{C}} c \in A$.

Let $(A, \mathbf{B})$ and $(A', \mathbf{B}')$ be two GT-algebras. Following [7] we say that a homomorphism $\varphi \in \mathrm{Hom}_R(A, A')$ is a GT-homomorphism, if

† To avoid confusion with complex conjugation we use $*$ instead of the traditional $^-$, [1].

H1. $\varphi(b^*) = \varphi(b)^*, b \in \mathbf{B}$;

H2. $\varphi$ is a homomorphism of $R$-algebras;

H3. For each $b \in \mathbf{B}$ there exist $b' \in \mathbf{B}'$ and $r_b \in R$ such that $r_b \neq 0$ and $\varphi(b) = r_b b'$.

If $(A, \mathbf{B})$ is real, then we require, in addition, $r_b > 0$ for each $b \in \mathbf{B}$.

If $\varphi$ is a bijection, then we say that $(A, \mathbf{B})$ and $(A', \mathbf{B}')$ are GT-isomorphic (or $(A', \mathbf{B}')$ is a rescaling of $(A, \mathbf{B})$). It should be mentioned that in this case all factors $r_b, b \in \mathbf{B}$, are invertible elements of $R$.

A subset $\mathbf{D} \subset \mathbf{B}$ is said to be **closed** (or **a table**) subset if the $R$-submodule $\langle c \rangle_{c \in \mathbf{D}}$ is a GT-algebra with distinguished basis $\mathbf{D}$.

In what follows the notation $\mathbf{D} \leq \mathbf{B}$ will mean that $\mathbf{D}$ is a closed subset of $\mathbf{B}$. A routine check shows that the intersection of two closed subsets is a closed subset as well. This justifies the following definition. Given $b \in \mathbf{B}$, we define $\mathbf{B}_b$ as the minimal closed subset containing $b$. An element $b \in \mathbf{B}$ will be called **faithful** (see [1]) if $\mathbf{B}_b = \mathbf{B}$. We say that $(A, \mathbf{B})$ is **primitive** if all non-identity elements of $\mathbf{B}$ are faithful. An element $b \in \mathbf{B}$ is called **real** (or **symmetric**), if $b^* = b$, [1].

*Examples:* Let $(X; \mathcal{G})$ be a **homogeneous coherent configuration**, [13] (an **association scheme** in [6]). Then its Bose–Mesner algebra is an example of a standard GT-algebra over $\mathbb{Z}$. All structure constants of the Bose–Mesner algebra are non-negative, so it is also a real algebra (one can use the equivalent notion of cell algebras introduced in [20]).

Let $(G; X)$ be a transitive permutation group. Then it acts naturally on the set $X^2$. The orbits of this action, called 2-orbits in [12] (orbitals in [11]), form a homogeneous coherent configuration. Its Bose–Mesner algebra coincides with the centralizer ring of the corresponding permutation representation of $G$. As a GT-algebra it is isomorphic to the Hecke algebra $H_{\mathbb{Z}}(G; H)$, where $H = G_x$ is a point stabilizer.

If $G$ is a semidirect product of $H$ by $\mathrm{Inn}(H)$ which acts on the group $H$ by the following rule:

$$x^{(\varphi, h)} = x^{\varphi} h, \quad \varphi \in \mathrm{Inn}(H), \quad x, h \in H,$$

then the correponding Bose–Mesner algebra is isomorphic (as a standard GT-algebra) to the centre of the group algebra $\mathbb{Z}H$ where the distiguished basis is formed by the characteristic functions of the conjugacy classes of $H$.

The ring of group characters with the irreducible ones as a distinguished basis is another example of a GT-algebra. The distinguished basis is not standard unless the underlying group is abelian, but it always may be rescaled into the standard one. We refer the reader to [1] for more details.

All previous examples were algebras over $\mathbb{Z}$. The generic Hecke–Iwahori algebras of finite Coxeter groups (see [14] for the details) give examples of standard GT-algebras defined over the ring of integral polynomials.

The paper is organized as follows. Section 2 contains basic facts about GT-algebras.

Section 3 deals with representation theory of GT-algebras. The theory of feasible traces developed by D. G. Higman is applied in order to obtain orthogonality relations for irreducible characters. In this section we also define the **Frame number** of a GT-algebra which is a direct generalization of the well-known numerical invariant of the Bose–Mesner algebra of a homogeneous coherent configuration. We show that the Frame number of a GT-algebra always belongs to the ring that contains the structure constants of the GT-algebra.

As a consequence of the developed theory we obtain the following theorems.

THEOREM 1.1: *Let $(A, \mathbf{B})$ be the Bose–Mesner algebra (over $\mathbb{Z}$) of a commutative association scheme $(X;\mathcal{G})$ with $d$ classes. Let $v_0 = 1, v_1, ..., v_d$ and $m_0 = 1, m_1, ..., m_d$ be its valencies and covalencies. Then $\mathbb{F}_p \otimes_{\mathbb{Z}} A$ is semisimple iff*

$$p \not| |X|^{d+1} \frac{v_1 \cdot \ldots \cdot v_d}{m_1 \cdot \ldots \cdot m_d}.$$

THEOREM 1.2: *Let $(X;\mathcal{G})$ be a homogeneous coherent configuration on a prime number of points. Assume that there exists $k$ such that $|g| = k$ for each $g \in \mathcal{G}, g \neq 1$. Then*†

(i) *The Bose–Mesner algebra of $(X;\mathcal{G})$ is commutative (i.e., $(X;\mathcal{G})$ is an association scheme in the sense of [6]).*

(ii) *All non-trivial covalencies of $(X;\mathcal{G})$ are equal to $k$.*

Section 4 is devoted to non-singular real GT-algebras. For real GT-algebras we show that every distinguished basis may be rescaled into a standard one which is unique. We also extend the theory of table subsets and generalize to the non-commutative case well-known properties of table algebras.

Integral GT-algebras with a standard distinguished basis are studied in Section 5. We study the properties of primitive integral standard GT-algebras. The main result of this section generalizes the well-known properties of primitive permutation groups (see [19]).

† Here $|g|$ is the valency of a relation $g \in G$.

THEOREM 1.3: *Let $(A, \mathbf{B})$ be an integral standard GT-algebra.*† *Assume that the elements of $\mathbf{B}$ are numbered in non-decreasing order of their degrees: $|b_1| = 1 \le |b_2| \le \cdots \le |b_d|$. Let $\pi(\mathbf{B}) = \bigcup_{i=2}^{i=d} \pi(|b_i|)$, where $\pi(m)$ denotes the set of all prime divisors of $m$.*

*If $(A, \mathbf{B})$ is primitive, then:*

(i) *If $|b_2| = 1$, then $(A, \mathbf{B})$ is a group algebra of a group of prime order.*
(ii) $\forall_{p \in \pi(\mathbf{B})} (p \le |b_2|)$.
(iii) *If $|b_2| > 1$ and $\gcd(|b_i|, |b_j|) = 1$ for some $1 < i, j$, then there exists $1 < k$ such that $|b_k| \mid |b_i||b_j|$ and $|b_k| > \max(|b_i|, |b_j|)$.*
(iv) *If $|b_2|$ is prime, then $|b_2|^2 \not| |b_j|$ for each $j \ge 1$.*
(v) *If $|b_2| > 1$, then $|b_i| \le |b_{i-1}|(|b_2| - 1)$ for each $2 \le i$.*
(vi) *If $|b_2| > 1$, then $\gcd(|b_i|, |b_d|) \ne 1$ for each $2 \le i$.*
(vii) *If $|b_2| = 2$, then $|b_i| = 2$ for all $1 \le i \le d$ and $(A, \mathbf{B})$ is a subalgebra of the group algebra $\mathbb{Z}C_p$ with $\mathbf{B} = \{1\} \cup \{g^i + g^{-i}\}_{0<i<p/2}$ (here $C_p$ is a cyclic group of prime order generated by $g \in C_p$).*

## 2. Basic properties

We start with the following claim:

PROPOSITION 2.1: *Let $(A, \mathbf{B})$ be a GT-algebra. Then for arbitrary $x, y, z \in A$ the following equalities hold:*

(i) $(x, yz) = (xz^*, y)$,
(ii) $(xy, z) = (y, x^*z)$.

*Proof:* (i) $(x, yz) = t(xz^*y^*) = (xz^*, y)$.
(ii) $(xy, z) = t(xyz^*) = t(yz^*x) = (y, x^*z)$. ∎

PROPOSITION 2.2: *Let $(A, \mathbf{B})$ be a GT-algebra. Then the following hold for each $a, b, c, d \in \mathbf{B}$:*

(i) $\sum_{t\in\mathbf{B}} \lambda_{abt}\lambda_{tcd} = \sum_{t\in\mathbf{B}} \lambda_{atd}\lambda_{bct}$;
(ii) $|1| = 1$ *and* $|b| = |b^*|$;
(iii) $\lambda_{abc} = \lambda_{b^*a^*c^*}$;
(iv) $\lambda_{abc}|c| = \lambda_{c^*ab^*}|b| = \lambda_{cb^*a}|a|$, *and* $\lambda_{bb^*c} = \lambda_{bb^*c^*}$;
(v) $\sum_{x\in\mathbf{B}} \lambda_{abx}^2|x| = \sum_{x\in\mathbf{B}} \lambda_{a^*ax}\lambda_{bb^*x}|x|$;
(vi) *if $(A, \mathbf{B})$ is non-singular, then* $\lambda_{aba} = \lambda_{ab^*a} = \lambda_{ba^*a^*} = \lambda_{b^*a^*a^*}$.

*Proof:* (i) is a direct consequence of the associative law.

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† Integral GT-algebras and their degrees are defined in Section 5.

(ii) follows from T3 and the definition of $|b|$.

(iii) is a consequence of the fact that $x \to x^*$ is an antiautomorphism.

(iv) By the definition of the bilinear form, $(ab, c) = \lambda_{abc}|c|$. By Proposition 2.1

$$(ab, c) = (a, cb^*) = (cb^*, a) = \lambda_{cb^*a}|a|,$$
$$(ab, c) = (b, a^*c) = (a^*c, b) = \lambda_{a^*cb}|b|.$$

Taking into account that $\lambda_{a^*cb} = \lambda_{c^*ab^*}$ and $|b| = |b^*|$, we obtain $\lambda_{a^*cb}|b| = \lambda_{c^*ab^*}|b|$.

The equality $\lambda_{bb^*c} = \lambda_{bb^*c^*}$ follows from the fact that $(bb^*)^* = bb^*$.

(v) As follows from (1),

$$(ab, ab) = \sum_{c\in\mathbf{B}} \lambda_{abc}^2|c|.$$

On the other hand, by Proposition 2.1

$$(ab, ab) = (bb^*, a^*a) = \sum_{c\in\mathbf{B}} \lambda_{bb^*c}\lambda_{a^*ac}|c|,$$

as desired.

(vi) is a direct consequence of (iv). ∎

PROPOSITION 2.3: *Let $(A, \mathbf{B})$ be a standard non-singular GT-algebra. Then:*

(i) $\forall_{a,b\in\mathbf{B}}\ \sum_{x\in\mathbf{B}} \lambda_{axb} = \sum_{x\in\mathbf{B}} \lambda_{xab} = |a|$.

(ii) $\forall_{a\in A}\ a\mathbf{B}^+ = \mathbf{B}^+a = |a|\mathbf{B}^+$.

(iii) *If $(A, \mathbf{B})$ is real, then for each $\mathbf{C} \subset \mathbf{B}$ and $a, b \in \mathbf{B}$ we have $\sum_{c\in\mathbf{C}} \lambda_{acb} \le |a|$.*

*Proof:* (i) Multiplying the sum $\sum_{x\in\mathbf{B}} \lambda_{axb}$ by $|b|$ we obtain

$$\sum_{x\in\mathbf{B}} \lambda_{axb}|b| = \sum_{x\in\mathbf{B}} \lambda_{b^*ax^*}|x|.$$

Since the map $x \mapsto |x|$ is a homomorphism, the right hand side is equal to $|a||b^*| = |a||b|$. Cancelling by $|b|$ we obtain $\sum_{x\in\mathbf{B}} \lambda_{axb} = |a|$.

In order to prove the second equality we apply $^*$ to the first one:

$$\sum_{x\in\mathbf{B}} \lambda_{xab} = \sum_{x\in\mathbf{B}} \lambda_{a^*x^*b^*} = |a^*| = |a|.$$

(ii) is a direct consequence of (i).

The last part of our claim is a direct consequence of (i). ∎

## 3. The representation theory of non-singular GT-algebras

In this section we assume that $R$ is a field and $\mathbf{B}$ is non-singular.

A linear function $f\colon A \to R$ is called **a feasible trace** [13] if it satisfies the identity $f(xy) = f(yx)$, $x, y \in A$. According to the axioms of GT-algebras the function $t$, defined in the introduction, is a feasible trace. So we may apply the representation theory of algebras with feasible traces which was created by D. G. Higman in [13]. Although this theory was developed under the assumption that $\mathsf{char}(R) = 0$, it remains valid if we require separability of the extension $\bar{R}/R$, where $\bar{R}$ is the algebraic closure of $R$. This condition always holds if $\mathsf{char}(R) = 0$ or $R$ is finite.

For every $a \in A$ we define a $\mathbf{B} \times \mathbf{B}$ matrix $M_a$ as the matrix of the linear operator $x \mapsto ax$, $x \in A$ in the basis $\mathbf{B}$. The map $a \mapsto M_a$ is the left regular representation of $A$. The character of this representation will be denoted by reg( ). Clearly, reg( ) is a feasible trace on $A$.

It is easy to see that $(M_a)_{bc} = \lambda_{abc}$ and $\mathrm{reg}(b) = \sum_{x\in\mathbf{B}} \lambda_{bxx}$, $b \in \mathbf{B}$. Using this character one can define the Killing form $K(a,b) = \mathrm{reg}(ab)$. By $D$ we denote the diagonal matrix defined as follows: $D_{bb} = |b|$, $b \in \mathbf{B}$.

PROPOSITION 3.1: (i) $M_{a^*} = DM_a^T D^{-1}$ *for each* $a \in \mathbf{B}$.
(ii) $\mathrm{reg}(a) = \mathrm{reg}(a^*)$ *for each* $a \in \mathbf{B}$.

*Proof:* (i) We just compute the $(b,c)$-entries of both sides.

$$\begin{aligned}(M_{a^*})_{bc} &= \lambda_{a^*bc} = \lambda_{c^*a^*b^*}|b||c|^{-1} = |b||c|^{-1}\lambda_{acb}\\ &= |b||c|^{-1}(M_a^T)_{bc} = (DM_a^TD^{-1})_{bc}.\end{aligned}$$

(We used Propositon 2.2, which says that $\lambda_{acb}|b| = \lambda_{b^*ac^*}|c^*| = \lambda_{a^*bc}|c|$, implying $|b||c|^{-1}\lambda_{acb} = \lambda_{a^*bc}$.)
(ii) is a direct consequence of (i). ∎

The following result is well-known.

THEOREM 3.2: *Let $A$ be an associative algebra over the field $R$. If its Killing form $K$ is non-degenerate, then $A$ is separable.*

In what follows we set

$$\Delta := \sum_{b\in\mathbf{B}} \frac{1}{|b|} b^* b.$$

PROPOSITION 3.3: *Let $f\colon A \to R$ be a feasible trace. Then:*
(i) $f^*(x) := f(x^*)$ *is a feasible trace;*

(ii) $z_f := \sum_{b\in \mathbf{B}} \frac{f(b^*)}{|b^*|} b \in Z(A)$.

*Proof:* Part (i) is trivial.

(ii) Take an arbitrary $a \in \mathbf{B}$ and consider the commutator:

$$[z_f, a] = \sum_{b\in \mathbf{B}} \frac{f(b^*)}{|b^*|}(ab - ba) = \sum_{b,c\in \mathbf{B}} \frac{f(b^*)}{|b^*|}(\lambda_{abc} - \lambda_{bac})c.$$

By Proposition 2.2

$$\lambda_{abc}|c| = \lambda_{c^*ab^*}|b^*|, \quad \lambda_{bac}|c| = \lambda_{c^*ba^*}|a^*| = \lambda_{ac^*b^*}|b^*|.$$

Therefore

$$\frac{1}{|b^*|}(\lambda_{abc} - \lambda_{bac}) = \frac{1}{|c|}(\lambda_{c^*ab^*} - \lambda_{ac^*b^*}),$$

implying that

$$\begin{aligned}[z_f, a] &= \sum_{c\in \mathbf{B}} \frac{1}{|c|} \sum_{b\in \mathbf{B}} (\lambda_{c^*ab^*} f(b^*) - \lambda_{ac^*b^*} f(b^*)) \\ &= \sum_{c\in \mathbf{B}} \frac{1}{|c|}(f(c^*a) - f(ac^*)).\end{aligned}$$

The latter equality shows that $z_f$ belongs to the centre $Z(A)$. ∎

PROPOSITION 3.4: *The following properties hold:*

(i) $\Delta = \sum_{b\in \mathbf{B}} \frac{\mathrm{reg}(b^*)}{|b|} b$.

(ii) $\Delta \in Z(A)$.

(iii) *Let $M_K, M_\Delta$ be the matrices of $K$ and $\Delta$ in the basis $\mathbf{B}$. Then*

$$M_K \cdot T \cdot D^{-1} = M_\Delta,$$

*where $T$ is the matrix of the permutation $b \mapsto b^*$ in the basis $\mathbf{B}$.*

*Proof:* (i)

$$\begin{aligned}\Delta &= \sum_{b\in \mathbf{B}} \frac{b^*b}{|b|} = \sum_{b\in \mathbf{B}} \sum_{c\in \mathbf{B}} \frac{\lambda_{b^*bc}}{|b|} c \\ &= \sum_{c\in \mathbf{B}} \left( \sum_{b\in \mathbf{B}} \frac{\lambda_{b^*bc}}{|b|} \right) c = \sum_{c\in \mathbf{B}} \frac{1}{|c|} \left( \sum_{b\in \mathbf{B}} \lambda_{c^*b^*b^*} \right) c = \sum_{c\in \mathbf{B}} \frac{\mathrm{reg}(c^*)}{|c|} c.\end{aligned}$$

Part (ii) follows immediately from Proposition 3.3, since reg is a feasible trace.

(iii) Part (i) of our claim implies that

$$M_\Delta = \sum_{c\in\mathbf{B}} \frac{\mathrm{reg}(c^*)}{|c|} M_c.$$

Therefore

$$(M_\Delta)_{ab} = \sum_{c\in\mathbf{B}} \frac{\mathrm{reg}(c^*)}{|c|} \lambda_{cab}.$$

By Proposition 2.2, $\lambda_{cab}|b| = \lambda_{b^*ca^*}|a| = \lambda_{ab^*c^*}|c|$, whence

$$(M_\Delta)_{ab} = \sum_{c\in\mathbf{B}} \frac{\mathrm{reg}(c^*)}{|b|} \lambda_{ab^*c^*} = \frac{1}{|b|} K(a, b^*). \qquad \blacksquare$$

As a direct consequence we obtain the following corollary:

COROLLARY 3.5: *Assume that $(A, \mathbf{B})$ is a GT-algebra with non-singular $\mathbf{B}$. If $\Delta = \sum_{b\in\mathbf{B}} \frac{b^*b}{|b|}$ is invertible, then $A$ is semisimple.*

3.1 ORTHOGONALITY RELATIONS FOR SEMISIMPLE GT-ALGEBRAS. In this subsection we assume that $(A, \mathbf{B})$ is semisimple and $\mathbf{B}$ is non-singular. Let $\bar{R}$ be the algebraic closure of $R$ and $A_{\bar{R}} = \bar{R} \otimes_R A$. Here and later on we shall assume that the extension $\bar{R}/R$ is separable. If $\mathsf{char}(R) = 0$ or $R$ is finite, then this condition is always satisfied. In particular, the Bose–Mesner algebra of a homogeneous coherent configuration, defined over $\mathbb{Q}$, satisfies the condition $\mathsf{char}(R) = 0$, and, therefore, we may apply to these algebras all the results of this section.

Let $\rho^1, \ldots, \rho^r$ be a complete set of pairwise non-isomorphic irreducible representations of $A_{\bar{R}}$ and $\chi_i = \mathrm{tr}(\rho^i)$ be their characters. Then $A_{\bar{R}} \cong \bigoplus_{i=1}^{i=r} M_{m_i}(\bar{R})$, where $m_i = \chi_i(\mathbf{1})$. Clearly $\sum_{i=1}^{i=r} m_i^2 = |\mathbf{B}|$.

According to [13], $t = \sum_{i=1}^{i=r} z_i\chi_i$, where $z_i \in \bar{R}$ are feasible multiplicities. Since $t(xy)$ is a non-degenerate bilinear form, all $z_i$'s are non-zeroes.

Denote by $\rho^i_{jk}(b)$, $1 \le j,k \le m_i$, $1 \le i \le r$ the $(j,k)$-entry of the $m_i \times m_i$-matrix $\rho^i(b)$, $b \in \mathbf{B}$. Let $\Lambda$ be the set of all triples $(i,j,k)$, $1 \le i \le r, 1 \le j \le m_i$, $1 \le k \le m_i$.

For an arbitrary triple $\lambda = (i,j,k) \in \Lambda$ we set:

$$m_\lambda := m_i, \quad \rho_\lambda(b) := \rho^i_{jk}(b); \quad \lambda^t := (i,k,j); \quad z_\lambda := z_i.$$

(Here we use the notation of [13].)

If $\sigma \in \mathrm{Gal}(\bar{R}/R)$ and $\rho^i\colon A \to M_m(\bar{R})$ is an irreducible representation, then the map $\rho^{i^\sigma}\colon A \to M_m(\bar{R})$ defined by

$$\rho^{i^\sigma}(b) = \rho^i(b)^\sigma, \quad b \in \mathbf{B}$$

is an irreducible representation of $A_{\bar{R}}$ as well. Thus $\mathrm{Gal}(\bar{R}/R)$ acts as a permutation group on the set $\{\rho^1, \ldots, \rho^r\}$ of the irreducible representation of $A$.

THEOREM 3.6 (Section 5 of [13])):

(i) $\sum_{b\in\mathbf{B}} \frac{1}{|b|}\rho_\lambda(b^*)\rho_\mu(b) = \delta_{\lambda\mu^t} z_\lambda^{-1}$, $\lambda, \mu \in \Lambda$ ((5.3), [13]);

(ii) $\sum_{\lambda\in\Lambda} z_\lambda \rho_\lambda(b)\rho_{\lambda^t}(c) = \delta_{bc^*}|c|, b, c \in \mathbf{B}$ ((5.5), [13]);

(iii) $\sum_{b\in\mathbf{B}} \frac{1}{|b|}\chi_i(b^*)\chi_j(b) = \delta_{ij}\frac{m_i}{z_i}$, $i = 1, \ldots, r$ ((5.4), [13]);

(iv) *the elements*

$$J_{\chi_s} = z_s \sum_{b\in\mathbf{B}} \frac{\chi_s(b^*)}{|b^*|} b, \quad s = 1, \ldots, r \quad ((5.7),\ [13])$$

*are the minimal central idempotents of* $A_{\bar{R}}$.

Let $U$ be a $\Lambda \times \mathbf{B}$-matrix entries of which are defined as follows:

$$U_{\lambda,b} = \rho_\lambda(b).$$

Denote by $Z$ the $\Lambda \times \Lambda$ diagonal matrix with the entries $Z_{\lambda\lambda} = z_\lambda$ and by $S$ the $\Lambda \times \Lambda$-matrix of the permutation $\lambda \mapsto \lambda^t$.

LEMMA 3.7: *Let $(A, \mathbf{B})$ be a semisimple GT-algebra with a non-singular distinguished basis $\mathbf{B}$. Assume that there exists a UFD*† *$R_0 \subset R$ such that $R$ is a field of fractions of $R_0$ and $\lambda_{abc} \in R_0$ for each $a, b, c \in \mathbf{B}$. Then:*

(i) *for each $a, b \in \mathbf{B}$, $(U^T \cdot S \cdot U)_{ab} = \tau(ab)$, where $\tau(x) = \sum_{i=1}^{i=r}\chi_i(x), x \in A_{\bar{R}}$;*

(ii) *$\tau(ab) \in R_0$ for each $a, b \in \mathbf{B}$;*

(iii)

$$\mathcal{F}(A, \mathbf{B}) := \prod_{b\in\mathbf{B}} |b| \cdot \prod_{\lambda\in\Lambda} \frac{1}{z_\lambda} \in R_0; \tag{2}$$

(iv) *if $(A, \mathbf{B})$ is standard, then $|\mathbf{B}^+|^{-2}\mathcal{F}(A, \mathbf{B}) \in R_0$.*

*Proof:* (i)

$$(U^T \cdot S \cdot U)_{ab} = \sum_{\lambda\in\Lambda} \rho_\lambda(a)\rho_{\lambda^t}(b) = \sum_{i=1}^{i=r}\sum_{j=1}^{j=m_i}\sum_{k=1}^{k=m_i} \rho^i_{jk}(a)\rho^i_{kj}(b) = \sum_{i=1}^{i=r}\chi_i(ab) = \tau(ab).$$

(ii) Since $ab = \sum_{c\in\mathbf{B}} \lambda_{abc}c$, $\lambda_{abc} \in R_0$, it is sufficient to show that $\tau(a) \in R_0$ for each $a \in \mathbf{B}$.

---

† UFD is an abbreviation of a "unique factorization domain".

For each $a \in \mathbf{B}$, $\chi_i(a)$ belongs to the integral closure $\bar{R}_0$ of $R_0$ in $\bar{R}$. Therefore, $\tau(a) \in \bar{R}_0$.

On the other hand, for each $\sigma \in \mathrm{Gal}(\bar{R}/R)$ we have

$$\tau(a)^\sigma = \sum_{i=1}^{i=r} (\chi_i(a))^\sigma = \sum_{i=1}^{i=r} \chi_{i^\sigma}(a) = \sum_{i=1}^{i=r} \chi_i(a) = \tau(a).$$

Therefore, $\tau(a) \in R$. Since $R_0$ is a UFD, $\bar{R}_0 \cap R = R_0$ implying $\tau(a) \in R_0$.

(iii) According to Theorem 3.6, part (i),

$$U \cdot T \cdot D^{-1} U^T = Z^{-1} \cdot S.$$

Therefore,

$$\det(U)^2 \det(T) \prod_{b \in \mathbf{B}} |b|^{-1} = \det(S) \prod_{\lambda \in \Lambda} z_\lambda^{-1}.$$

Since $S$ and $T$ are permutation matrices,

$$\det(U)^2 = \pm \prod_{\lambda \in \Lambda} z_\lambda^{-1} \cdot \prod_{b \in \mathbf{B}} |b|.$$

On the other hand,

$$\det(U)^2 \det(S) = \det(\tau(ab))_{a,b \in \mathbf{B}} \in R_0.$$

Therefore,

$$\mathcal{F}(A, \mathbf{B}) = \prod_{b \in \mathbf{B}} |b| \prod_{\lambda \in \Lambda} z_\lambda^{-1} \in R_0.$$

(iv) Consider the following basis of $A$:

$$e_b = \begin{cases} \mathbf{B}^+ & \text{if } b = \mathbf{1}, \\ b & \text{if } b \in \mathbf{B} \setminus \{\mathbf{1}\}. \end{cases}$$

The determinant of the transition matrix from the basis $\mathbf{B}$ to the above one is equal to 1. Therefore,

$$\det(\tau(ab))_{a,b \in \mathbf{B}} = \det(\tau(e_a e_b))_{a,b \in \mathbf{B}}.$$

Since $b \mapsto |b|$ is an irreducible representation, it coincides with some $\rho^i$. Without loss of generality we may assume that $\rho^1(b) = |b|$, $b \in \mathbf{B}$. By Proposition 2.3 (ii), $b\mathbf{B}^+ = \mathbf{B}^+ b = |b|\mathbf{B}^+$. Therefore,

$$\tau(e_b e_\mathbf{1}) = \begin{cases} |b|\tau(e_\mathbf{1}), & b \neq \mathbf{1}, \\ |\mathbf{B}^+|\tau(e_\mathbf{1}), & b = \mathbf{1}. \end{cases}$$

By Theorem 3.6 (iii)

$$\sum_{b\in\mathbf{B}}\frac{\chi_1(b^*)\chi_i(b)}{|b|}=0,$$

whence $\sum_{b\in\mathbf{B}}\chi_i(b) = 0$ for each $i>1$. Therefore, $\tau(\mathbf{B}^+) = |\mathbf{B}^+|$, whence $\tau(e_1e_1) = |\mathbf{B}^+|\tau(e_1) = |\mathbf{B}^+|^2$. Thus the matrix of the form $\tau$ in the basis $\{e_b\}_{b\in\mathbf{B}}$ has the following form:

$$\begin{pmatrix} |\mathbf{B}^+|^2 & \dots & |b||\mathbf{B}^+| & \dots \\ . & * & \dots & * \\ |b||\mathbf{B}^+| & . & \dots & . \\ . & * & \dots & * \end{pmatrix}$$

where $*$ denotes the elements $\tau(e_be_a), a,b\in\mathbf{B}\setminus\{1\}$. Since $\tau(e_be_a)\in R_0$ for all $a,b\in\mathbf{B}\setminus\{1\}$, the determinant of the above matrix is equal to $|\mathbf{B}|^2r, r\in R_0$. ∎

*Remark:* If $(A,\mathbf{B})$ is the Bose–Mesner algebra (over $\mathbb{C}$) of a homogeneous coherent configuration $(X;\mathcal{G})$, then

$$z_i=\frac{f_i}{|X|},$$

where $f_i$ is the multiplicity of the representation $\rho_i$ in the decomposition of the standard module $\mathbb{C}X$ (see [22], [20]).† In this case

$$\mathcal{F}(A,\mathbf{B})=|X|^d\frac{\prod_{b\in\mathbf{B}}|b|}{\prod_{i=1}^{i=r}f_i^{m_i^2}},$$

where $d=\dim(A)$ and $|b|$ is the valency of $b\in\mathbf{B}$. The number

$$|X|^{d-2}\frac{\prod_{b\in\mathbf{B}}|b|}{\prod_{i=1}^{i=r}f_i^{m_i^2}}$$

is known as the Frame quotient of the configuration [21]. For this reason we shall call $\mathcal{F}(A,\mathbf{B})$ **the Frame number** of the GT-algebra $(A,\mathbf{B})$. It was shown in [20] that the Frame quotient of an arbitrary homogeneous coherent configuration is a rational integer. Thus part (iii) of the above Lemma is a direct generalization of this fact.

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† The multiplicities $f_i$ are called covalencies if $A$ is commutative.

LEMMA 3.8: *Let $(A, \mathbf{B})$ be a semisimple algebra with non-singular distinguished basis. Denote $m_i := \chi_i(\mathbf{1})$, $i = 1, \dots, r$. Then:*

(i) $\Delta = \sum_{i=1}^{i=r} \frac{m_i}{z_i} J_{\chi_i}$.

(ii) $\prod_{i=1}^{i=r} \left(\Delta - \frac{m_i}{z_i}\mathbf{1}\right) = 0$.

(iii) *The characteristic polynomial of $M_\Delta$ is equal to $\prod_{i=1}^{i=r}(x - \frac{m_i}{z_i})^{m_i^2}$.*

(iv) $\det(M_K) = \pm \prod_{b\in\mathbf{B}} |b| \prod_{i=1}^{r} \left(\frac{m_i}{z_i}\right)^{m_i^2}$.

*Proof:* (i) Since $\mathrm{reg}\colon A \to R$ is the regular character of a semisimple algebra, $\mathrm{reg} = \sum_{i=1}^{i=r} m_i\chi_i$. Therefore

$$\Delta = \sum_{b\in\mathbf{B}} \frac{\mathrm{reg}(b^*)}{|b|} b = \sum_{b\in\mathbf{B}} \sum_{i=1}^{i=r} \frac{m_i\chi_i(b^*)}{|b|} b = \sum_{i=1}^{i=r} \frac{m_i}{z_i} J_{\chi_i}.$$

(ii) It follows from the previous part that

$$(\Delta - \frac{m_i}{z_i}\mathbf{1})J_{\chi_i} = 0.$$

Since $J_{\chi_i}$, $1 \le i \le r$ are primitive central idempotents of $A$,

$$\prod_{i=1}^{i=r} \left(\Delta - \frac{m_i}{z_i}\mathbf{1}\right) J_{\chi_j} = 0 \quad \text{for each } 1 \le j \le r.$$

To finish the proof it is enough to note that $\mathbf{1} = \sum_{i=1}^{i=r} J_{\chi_j}$.

(iii) follows easily from (i) and the fact that $m_j^2$ is the rank of $J_{\chi_j}$ in the left regular representation.

(iv) By Proposition 3.4 $\det(T)\det(M_K) = \det(D)\det(M_\Delta)$, whence $\det(M_K) = \pm \prod_{b\in\mathbf{B}} |b| \det(M_\Delta)$. By the previous part

$$\det(M_\Delta) = \pm \prod_{i=1}^{r} \left(\frac{m_i}{z_i}\right)^{m_i^2}. \qquad \blacksquare$$

*Proof of Theorem 1.2:* Let $A$ be the Bose–Mesner $\mathbb{Z}$-algebra of $\mathcal{G}$ and $\mathbf{B} = \{B_g \mid b \in \mathcal{G}\}$ be its distinguished basis.

Let $m_0 = 1, m_1, \dots, m_r$ be the degrees of the irreducible complex representations of $A$ and $f_i$ be the multiplicity of the $i$-th irreducible representation in the decomposition of the standard module $\mathbb{C}X$. Denote $d = \dim(A)$. It follows from Proposition 3.4 that the eigenvalues of the matrix $kM_\Delta$ are algebraic integers.

On the other hand, the remark after Lemma 3.7 and Lemma 3.8 imply that the eigenvalues of $M_\Delta$ are $\frac{m_i|X|}{f_i}$, $i=1,\ldots,r$. Therefore

$$k\frac{m_i|X|}{f_i} \in \mathbb{Z}, \quad i=1,\ldots,r.$$

But $|X|$ is prime and $m_i f_i < |X|, k < |X|$. Therefore $\frac{km_i}{f_i} \in \mathbb{Z}$, or, equivalently,

$$\frac{f_i}{m_i} = \frac{k}{u_i}, \quad u_i \in \mathbb{N}, \quad i=1,\ldots,r.$$

Now we can write

$$|X|-1 = \sum_{i=1}^{r} m_i f_i = \sum_{i=1}^{r} \frac{k}{u_i} m_i^2.$$

On the other hand,

$$|X|-1 = k(d-1) = \sum_{i=1}^{r} k m_i^2.$$

Comparing both equalities we obtain $u_i = 1$ for each $i=1,\ldots,r$. Thus $f_i = m_i k$, $i=1,\ldots,r$. Now the Frame number of the algebra $(A,\mathbf{B})$ is equal to

$$\mathcal{F}(A,\mathbf{B}) = |X|^d \frac{1}{\prod_{i=1}^{r} m_i^{m_i^2}}.$$

Since $|X|$ is prime and strictly greater than $m_i$, the above number may be an integer only in one case: $m_i = 1$ for every $i=1,\ldots,r$. ∎

LEMMA 3.9: *Let $(A,\mathbf{B})$ be the Bose–Mesner algebra of a homogeneous coherent configuration of degree $n$ with $d$ classes. Denote by $G$ a $(d+1)\times(d+1)$-matrix whose entries are defined by the formula*

$$G_{ab} = \sum_{c\in\mathbf{B}} \lambda_{ab^*c}\left(\sum_{d\in\mathbf{B}} \lambda_{cdd}\right).$$

*Then*

$$\det(xG - nD) = \mathcal{F}(A,\mathbf{B}) \prod_{i=1}^{r} (m_i x - f_i)^{m_i^2},$$

*where $m_i$ is the degree of the $i$-th irreducible representation and $f_i$ is its multiplicity in the decomposition of the standard module.*

*Proof:* By definition $G_{ab} = K(a,b^*)$, or, equivalently, $G = M_K T$. By Proposition 3.4 (ii), $G = M_\Delta D$ and we have

$$\det(xD - G) = \det(xI - M_\Delta)\det(D) = \det(D)\prod_{i=1}^{r}\left(x - \frac{m_i}{z_i}\right)^{m_i^2}.$$

Now by substituting $x^{-1}$ instead of $x$ we obtain

$$\det(D - xG) = \det(D)\prod_{i=1}^{r}\left(1 - \frac{m_i}{z_i}x\right)^{m_i^2} \Leftrightarrow$$

$$\begin{aligned}\det(xG - D) &= \det(D)\prod_{i=1}^{r}\left(\frac{m_i}{z_i}x - 1\right)^{m_i^2}\\ &= \det(D)\prod_{i=1}^{r}\left(\frac{m_i}{z_i}\right)^{m_i^2}\prod_{i=1}^{r}\left(x - \frac{z_i}{m_i}\right)^{m_i^2}\\ &= \mathcal{F}(A,\mathbf{B})\prod_{i=1}^{r}m_i^{m_i^2}\prod_{i=1}^{r}\left(x - \frac{z_i}{m_i}\right)^{m_i^2}.\end{aligned}$$

After substitution of $z_i = f_i/n$ and $x/n$ instead of $x$ we obtain

$$\det\left(\frac{x}{n}G - D\right) = \mathcal{F}(A,\mathbf{B})\left(\prod_{i=1}^{r}m_i^{m_i^2}\right)\prod_{i=1}^{r}\left(\frac{x}{n} - \frac{f_i}{m_i}\frac{1}{n}\right)^{m_i^2} \Leftrightarrow$$

$$\det(xG - nD) = \mathcal{F}(A,\mathbf{B})\prod_{i=1}^{i=r}m_i^{m_i^2}\left(x - \frac{f_i}{m_i}\right)^{m_i^2},$$

as desired. ∎

Theorem 1.1 is a direct consequence of the following claim.

THEOREM 3.10: *Let $(A,\mathbf{B})$ be a commutative semisimple GT-algebra with non-singular $\mathbf{B}$ defined over the field $R$. Assume that $R$ contains a UFD $R_0$ such that $\lambda_{abc} \in R_0$ holds for all $a,b,c \in R_0$. Assume that $R$ is a field of fractions of $R_0$. Denote by $R_0\mathbf{B}$ the $R_0$-algebra generated by $\mathbf{B}$.*

*Let $I$ be a maximal ideal of $R_0$ and $F = R_0/I$. Assume, in addition, that the algebraic closure $\bar{F}$ is a separable extension of $F$. Then the $F$-algebra $A_F := (R_0\mathbf{B})\,/\,(I\mathbf{B})$ is semisimple if and only if the Frame number $\mathcal{F}(A,\mathbf{B})$ is non-zero modulo $I$.*

*Proof:* Since $\bar{F}/F$ is a separable extension, a finite dimensional commutative $F$-algebra is semisimple if and only if its Killing form is non-degenerate. The determinant of the Killing form of $A_F$ in the basis $\mathbf{B}$ is the residue class of the determinant of the Killing form of $R_0\mathbf{B}$ in the basis $\mathbf{B}$. By Lemma 3.8 the determinant of the Killing form is equal to

$$\pm\prod_{b\in\mathbf{B}}|b|\prod_{\lambda\in\Lambda}\frac{m_\lambda}{z_\lambda}.$$

Since $A$ is commutative, $m_\lambda = 1$, for each $\lambda \in \Lambda$. But in this case the above product is equal to $\mathcal{F}(A, \mathbf{B})$ up to a sign. ∎

3.2 THE REPRESENTATIONS OF REAL GT-ALGEBRAS. In this subsection we assume that $(A, \mathbf{B})$ is a non-singular real GT-algebra, i.e.,

$$\forall_{a,b,c\in\mathbf{B}} \quad \lambda_{abc} \in \mathbb{R}, \quad \lambda_{abc} \geq 0 \quad \text{and} \quad \lambda_{bb^*\mathbf{1}} > 0.$$

Since the extension $\mathbb{C}/\mathbb{R}$ is separable we may apply the results of the previous subsection for studying real GT-algebras. In what follows we shall use the following notation:

$$\mathbb{R}^{\geq 0} = \{r \in \mathbb{R} | \ r \geq 0\}, \quad \mathbb{R}^{>0} = \{r \in \mathbb{R} | \ r > 0\}.$$

THEOREM 3.11: *$A$ is a semisimple algebra.*

*Proof:* Let $J$ be the Jacobson radical of $A$. Since $^*$ is an antiautomorphism of $A$, $J^* = J$.

Assume that $J \neq \{0\}$. $J$ is nilpotent, hence there exists a minimal $m \in \mathbb{N}$, $m \geq 2$ such that $J^m = \{0\}$. Set $I = J^{m-1}$. By the choice of $m$, $I \neq \{0\}$ and $I^2 = \{0\}$. Since $J^* = J$, $I^* = I$. Take a non-zero $x \in I$, $x = \sum_{b\in\mathbf{B}} x_b b, x_b \in \mathbb{R}$. Then $x^* = \sum_{b\in\mathbf{B}} x_b b^* \in I$. Further, $xx^* \in I^2 = \{0\}$. On the other hand, the coefficient of $\mathbf{1}$ in the product $xx^*$ is equal to $\sum_{b\in\mathbf{B}} \lambda_{bb^*\mathbf{1}} x_b^2 > 0$, a contradiction. ∎

This claim implies that $A_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} A$ is isomorphic to a direct product of full matrix algebras

$$A_\mathbb{C} \cong \bigoplus_{i=1}^{i=r} M_{m_i}(\mathbb{C}),$$

where $m_i$, $1 \leq i \leq r$ are the degrees of the irreducible representations of $A_\mathbb{C}$. Computing the dimensions we obtain $\sum_{i=1}^{i=r} m_i^2 = |\mathbf{B}|$.

An algebra homomorphism $f\colon A \to \mathbb{R}$ is called **a degree homomorphism** if $f(b) > 0$ holds for all $b \in \mathbf{B}$.

PROPOSITION 3.12: *If there exists a degree homomorphism, then it is unique.*

*Proof:* Let $f, g$ be two degree homomorphisms, $f \neq g$. Then Theorem 3.6 (iii) implies

$$\sum_{b\in\mathbf{B}} \frac{f(b^*)g(b)}{|b|} = 0,$$

a contradiction. ∎

PROPOSITION 3.13: *For each $a \in \mathbf{B}$ we have $a\mathbf{B} = \mathbf{B}$.*

*Proof:* We have to show that for each $c \in \mathbf{B}$ there exists $b \in \mathbf{B}$ with $\lambda_{abc} \neq 0$. By Proposition 2.2 (v), $(c^*a, c^*a) \geq |a||c| > 0$. Therefore $\mathrm{Supp}(c^*a) \neq \emptyset$. Take an arbitrary $b \in \mathrm{Supp}(c^*a)$. Then $\lambda_{c^*ab} \neq 0$, implying $\lambda_{ab^*c} \neq 0$, as desired. ∎

The proof of Theorem below is a direct generalization of the proof of Lemma 2.5 of [1].

THEOREM 3.14: *There exists a unique degree homomorphism $f: A \to \mathbb{C}$ such that $f(b) = f(b^*)$ holds for each $b \in \mathbf{B}$.*

*Proof:* Consider the element $\mathbf{B}^+ = \sum_{b\in\mathbf{B}} b$. Its matrix $M_{\mathbf{B}^+}$ in the left regular representation is equal to the sum $M_{\mathbf{B}^+} = \sum_{b\in\mathbf{B}} M_b$. By Proposition 3.13 all entries of $M_{\mathbf{B}^+}$ are positive real numbers. By the Perron–Frobenius theorem there exists a unique maximal eigenvalue, say $d$, of multiplicity 1. The corresponding eigenvector $z = \sum_{b\in\mathbf{B}} v_b b$ has positive coordinates $v_b$. It is easy to see that $zb$ is also an eigenvector of $M_{\mathbf{B}^+}$ with the same eigenvalue $d$. Therefore $zb = f(b)z$ for a suitable $f(b) \in \mathbb{C}$. Since all coefficients of $z$ are positive reals, $\mathrm{Supp}(zb) = \mathbf{B}$ and $f(b) \in \mathbb{R}^{>0}$. Thus there exists a function $f: \mathbf{B} \to \mathbb{R}^{>0}$ such that $z \cdot b = f(b)z$. Now we can write

$$f(b)f(a)z = z(ab) = z(\sum_{c\in\mathbf{B}} \lambda_{abc}c) = (\sum_{c\in\mathbf{B}} \lambda_{abc} f(c))z,$$

whence

$$f(a)f(b) = \sum_{c\in\mathbf{B}} \lambda_{abc} f(c),$$

proving that $f$ is a homomorphism. Since $f(b) > 0$ for each $b \in \mathbf{B}$, $f$ is a degree homomorphism. To finish the proof it is sufficient to show that $f(b) = f(b^*)$ holds for each $b \in \mathbf{B}$. Consider the map $g: b \mapsto f(b^*)$, $b \in \mathbf{B}$. Since $^*$ is an antiautomorphism and $f$ is a homomorphism into a commutative algebra, $g$ is also an algebra homomorphism. Since $g(b) = f(b^*) > 0$, $g$ is a degree homomorphism. By Proposition 3.12, $g = f$. ∎

Since every real GT-algebra has a degree homomorphism, one can always rescale it into the standard GT-algebra. If $f$ is the degree homomorphism of $(A, \mathbf{B})$, then the number $o(\mathbf{B}) = \sum_{b\in\mathbf{B}} f(b)^2/|b|$ is invariant under rescaling [1]. Following [1] we shall call it **the order** of $(A, \mathbf{B})$.

THEOREM 3.15: *Let $f$ be the degree homomorphism of a real non-singular GT-algebra $(A, \mathbf{B})$. Denote*

$$b^s = \frac{f(b)}{|b|} b \quad \textit{and} \quad \mathbf{B}^s = \{b^s \,|\, b \in \mathbf{B}\}.$$

*Then*

(i) *$(A, \mathbf{B}^s)$ is a standard real algebra isomorphic to $(A, \mathbf{B})$.*

(ii) *If $(A, \mathbf{B}')$ is a standard real algebra obtained by a rescaling from $(A, \mathbf{B})$, then $\mathbf{B}' = \mathbf{B}^s$.*

*Proof:* (i) Denote by $\lambda^s_{a^s b^s c^s}$, $a^s, b^s, c^s \in \mathbf{B}^s$ the structure constants of $A$ in the new basis $\mathbf{B}^s$. Since $b^s \mapsto f(b^s) = f^2(b)/|b| = \lambda^s_{b^s b^{s*} \mathbf{1}}$ is an algebra homomorphism, $(A, \mathbf{B}^s)$ is a standard real GT-algebra.

(ii) Since $\mathbf{B}'$ is a rescaling of $\mathbf{B}$, each $b' \in \mathbf{B}'$ is of the form $b' = \mu_b b$, where $\mu_b > 0$. Since $\mathbf{B}'$ is standard, the map $g$: $b' \mapsto \lambda_{b'(b^*)'\mathbf{1}} = \mu_b^2 |b|$ is an algebra homomorphism. Therefore $g(b) = \mu_b |b|$ is a degree homomorphism of $A$. By Proposition 3.12, $g(b) = f(b)$, whence $\mu_b = f(b)/|b|$ and the claim follows. ∎

## 4. Properties of real standard GT-algebras

In this section we assume that $(A, \mathbf{B})$ is a real standard and non-singular GT-algebra.

Let $x \in A$ be an arbitrary element. We define the following subsets:

$$L_x = \{b \in \mathbf{B} \,|\, bx = |b|x\}; \quad R_x = \{b \in \mathbf{B} \,|\, xb = |b|x\}; \quad S_x = L_x \cap R_x.$$

The following claim was proved in [5].

PROPOSITION 4.1: *Let $(A, \mathbf{B})$ be a real non-singular and standard GT-algebra. Then for every $x \in A$, $x \neq 0$, $L_x, R_x, S_x$ are closed subsets of $\mathbf{B}$.*

*Remark:* This Proposition was proved in [5] for combinatorial algebras which, in our new terminology, are real non-singular and standard GT-algebras.

Analogously, for each $\mathbf{A} \subset \mathbf{B}$ we define

$$L_{\mathbf{A}} = \{b \in \mathbf{B} \,|\, b\mathbf{A} \subset \mathbf{A}\}; \quad R_{\mathbf{A}} = \{b \in \mathbf{B} \,|\, \mathbf{A}b \subset \mathbf{A}\}; \quad S_{\mathbf{A}} = L_{\mathbf{A}} \cap R_{\mathbf{A}}.$$

The following statement connects these definitions with the previous ones.

PROPOSITION 4.2: *Let $\mathbf{A} \subset \mathbf{B}$ be an arbitrary subset. Denote $\mathbf{A}^+ = \sum_{a\in\mathbf{A}} a$. Then the following equalities hold:*

(i) $L_{\mathbf{A}} = L_{\mathbf{A}^+}$;

(ii) $R_{\mathbf{A}} = R_{\mathbf{A}^+}$;

(iii) $S_{\mathbf{A}} = S_{\mathbf{A}^+}$.

*Proof:* The third part of the claim is a direct consequence of the first and second parts. Part (ii) follows from (i) by applying the antiautomorphism $^*$. Thus we have to prove only the first part.

The inclusion $L_{\mathbf{A}^+} \subset L_{\mathbf{A}}$ is evident. Let us prove the reverse inclusion.

Take an arbitrary $b \in L_{\mathbf{A}}$. Then

$$b\mathbf{A}^+ = \sum_{a\in\mathbf{A}} \mu_a a, \tag{3}$$

where $\mu_a$, $a \in \mathbf{A}$ are non-negative real numbers. By Proposition 2.3 part (iii), $\mu_a \le |b|$ for all $a \in \mathbf{A}$. Applying $|\ |$ to both parts of (3) we obtain

$$|b||\mathbf{A}^+| = \sum_{a\in\mathbf{A}} \mu_a |a|.$$

Therefore $\mu_a = |b|$ for all $a \in \mathbf{A}$. Thus $L_{\mathbf{A}} \subset L_{\mathbf{A}^+}$. ∎

Given $a \in \mathbf{B}$, we associate **a graph** $\Gamma_a$ (see [10]) as follows:

$$V(\Gamma_a) = \mathbf{B}, \quad E(\Gamma_a) = \{(b,c) \mid \lambda_{abc} \neq 0\}.$$

Since $M_{a^*} = D(M_a^T)D^{-1}$, $\Gamma_{a^*} = \Gamma_a^T$, where $\Gamma_a^T$ is obtained from $\Gamma_a$ by inverting its edges.

The following fact is well-known in the theory of association schemes. To make the text self-contained we reproduce its proof.

PROPOSITION 4.3: *Let $a \in \mathbf{B}$ be an arbitrary element. Then the following conditions are equivalent:*

(i) *$a$ is faithful;*

(ii) *$\Gamma_a$ is strongly connected.*

*Proof:* (ii)⇒(i) Assume that $\Gamma_a$ is strongly connected. Then for each $b \in \mathbf{B}$ there exists a path $b_0 = 1, b_1, \ldots, b_m = b$ such that $(b_i, b_{i+1}) \in E(\Gamma_a)$. This is equivalent to $\lambda_{ab_ib_{i+1}} \neq 0$ for every $i = 0, \ldots, m-1$. Therefore $b \in \mathrm{Supp}(a^m) \subset \mathbf{B}_a$, implying $\mathbf{B}_a = \mathbf{B}$.

(i)$\Rightarrow$(ii) Assume that $\Gamma_a$ is not strongly connected, i.e., there exists a partition $\mathbf{B} = \mathbf{B}_0 \cup \mathbf{B}_1, \mathbf{B}_0 \neq \emptyset, \mathbf{B}_1 \neq \emptyset$ such that $\lambda_{abc} = 0$ for any $b \in \mathbf{B}_0$ and $c \in \mathbf{B}_1$. Then $\mathrm{Supp}(a\mathbf{B}_0) \subset \mathbf{B}_0$. By Proposition 4.2, $a \in L_{\mathbf{B}_0^+}$. Since $a$ is faithful, $\mathbf{B}_a = \mathbf{B}$. Therefore $L_{\mathbf{B}_0^+} = \mathbf{B}$, implying $\mathbf{B}_0 = \mathbf{B}$, a contradiction. ∎

PROPOSITION 4.4: *Let $(A, \mathbf{B})$ be a real GT-algebra with standard $\mathbf{B}$. Then for each $a \in \mathbf{B}$ and any $\mathbf{C} \subset \mathbf{B}$ the following two conditions are equivalent:*

(i) *$\mathrm{Supp}(a^*a) \subset L_{\mathbf{C}^+}$;*

(ii) *there exists $\mathbf{D} \subset \mathbf{B}$ such that $a\mathbf{C}^+ = |a|\mathbf{D}^+$.*

*Proof:* (i) $\Rightarrow$ (ii). $\mathrm{Supp}(a^*a) \subset L_{\mathbf{C}^+} \Leftrightarrow a^*a\mathbf{C}^+ = |a|^2\mathbf{C}^+ \Rightarrow (a^*a\mathbf{C}^+, \mathbf{C}^+) = |a|^2|\mathbf{C}^+| \Leftrightarrow (a\mathbf{C}^+, a\mathbf{C}^+) = |a|^2|\mathbf{C}^+|$. Denote $\mathbf{D} := \mathrm{Supp}(a\mathbf{C}^+)$. Then $a\mathbf{C}^+ = \sum_{d\in\mathbf{D}} \mu_d d$ and $|a|^2|\mathbf{C}| = (a\mathbf{C}^+, a\mathbf{C}^+) = \sum_{d\in\mathbf{D}} \mu_d^2|d|$. Since $\mu_d \leq |a|$, the right-hand side of the latter equality is less than or equal to $|a|\sum_{d\in\mathbf{D}} \mu_d|d| = |a|^2|\mathbf{C}|$. Therefore $\mu_d = |a|$ holds for all $d \in \mathbf{D}$.

(ii) $\Rightarrow$ (i). First we show that $a^*\mathbf{D} \subset \mathbf{C}$. Assume the contrary, i.e., there exist $d \in \mathbf{D}$, $f \notin \mathbf{C}$ such that $\lambda_{a^*df} \neq 0$. Then $\lambda_{afd} \neq 0$, implying that the coefficient of $d$ in $a(\{f\} \cup \mathbf{C})^+$ is equal to $|a| + \lambda_{afd} > |a|$, contrary to Proposition 2.3 (iii). Therefore $a^*\mathbf{D} \subset \mathbf{C} \Rightarrow a^*\mathbf{D}^+ = \sum_{c\in\mathbf{C}} \mu_c c, \mu_c \in \mathbb{R}^{\geq 0}$, implying $a^*a\mathbf{C} \subset \mathbf{C}$. Since all structure constants are nonnegative, and by Proposition 4.2, it follows that $\mathrm{Supp}(a^*a) \subset L_{\mathbf{C}} = L_{\mathbf{C}^+}$. ∎

4.1 CLOSED SUBSETS AND QUOTIENT ALGEBRAS. Let $(A, \mathbf{B})$ be a real non-singular GT-algebra. Without loss of generality we may assume that $\mathbf{B}$ is standard. We remind the reader that a subset $\mathbf{C} \subset \mathbf{B}$ is closed if the subspace spanned by $\mathbf{C}$ is a GT-algebra with a distinguished basis $\mathbf{C}$. It follows directly from the axioms of GT-algebra that $\mathbf{C}$ is closed if and only if $\mathbf{1} \in \mathbf{C}$, $\mathbf{C}^* = \mathbf{C}$ and $ab = \sum_{c\in\mathbf{C}} \lambda_{abc}c$ for each pair of $a, b \in \mathbf{C}$.

The following claim is an immediate consequence of the definition of "closed subset" and "real", and the axioms for a GT-algebra.

PROPOSITION 4.5: *A subset $\mathbf{A} \subset \mathbf{B}$, $\mathbf{A} \neq \emptyset$ is closed if and only if* $\mathrm{Supp}(ab^*) \subset \mathbf{B}$ *for all $a, b \in \mathbf{B}$.*

As usual we define **a right (left) C-coset** as an arbitrary set of the form $b\mathbf{C}$ (resp. $\mathbf{C}b$), $b \in \mathbf{B}$. A subset $\mathbf{C}b\mathbf{C}$, $b \in \mathbf{B}$ will be called **a double coset** with respect to a closed subset $\mathbf{C}$.

An element $a \in \mathbf{B}$ is called **linear** (or **thin**) if $aa^* = \lambda_{aa^*\mathbf{1}}\mathbf{1}$. Since $A$ is a finite-dimensional algebra, the above equality implies $a^*a = \lambda_{a^*a\mathbf{1}}\mathbf{1} = \lambda_{aa^*\mathbf{1}}\mathbf{1}$. The set of all linear elements will be denoted by $\mathsf{L}(\mathbf{B})$.

PROPOSITION 4.6: (i) $|b| \geq 1$ *and equality holds iff* $b$ *is linear.*

(ii) *The set* $\mathsf{L}(\mathbf{B}) \subset \mathbf{B}$ *of linear elements of* $\mathbf{B}$ *is a group.*

*Proof:* (i) We have that $bb^* = |b|\mathbf{1} + \sum_{c\in\mathbf{B}} \lambda_{bb^*c}c$. Applying $|\ |$ to both sides of this equality we obtain the first part of the claim.

(ii) Let $a, b \in \mathsf{L}(\mathbf{B})$ be two arbitrary elements. Since both $a$ and $b$ are invertible in $A$, $ab$ is an invertible element of $A$. Thus, we need to show that $b, a \in \mathsf{L}(\mathbf{B})$ implies $ab \in \mathsf{L}(\mathbf{B})$.

Indeed, by Proposition 2.2

$$\sum_{c\in\mathbf{B}} \lambda_{abc}^2|c| = \sum_{c\in\mathbf{B}} \lambda_{a^*ac}\lambda_{bb^*c}|c| = 1. \tag{4}$$

On the other hand,

$$\sum_{c\in\mathbf{B}} \lambda_{abc}|c| = |a||b| = 1. \tag{5}$$

Together with $|c| \geq 1$ this implies $\lambda_{abc} \leq 1$ for each $c \in \mathbf{B}$. Let $d$ be an element with the maximal value of $\lambda_{abd}$. Then (4) implies

$$\lambda_{abd} = \lambda_{abd}\left(\sum_{c\in\mathbf{B}} \lambda_{abc}|c|\right) \geq \sum_{c\in\mathbf{B}} \lambda_{abc}^2|c| = 1,$$

whence $\lambda_{abd} \geq 1$. Therefore $\lambda_{abd} = 1$. Combining this with (5) and $|d| \geq 1$ we obtain $ab = d$, $d \in \mathbf{B}$. ∎

*Remark:* A part of the claim below was proved in [22] for the Bose–Mesner algebras of homogeneous coherent configurations. The proof given in [22] uses the geometrical properties of homogeneous coherent configurations. So, here we give an independent proof, which uses only the axioms of GT-algebras.

PROPOSITION 4.7: *Let* $(A, \mathbf{B})$ *be a non-singular real GT-algebra with standard* $\mathbf{B}$ *and* $\mathbf{C} \leq \mathbf{B}$ *be an arbitrary closed subset. Then the following conditions hold for every* $b, d \in \mathbf{B}$*:*

(i) $\mathbf{C}b \cap \mathbf{C}d \neq \emptyset \Leftrightarrow \mathbf{C}b = \mathbf{C}d$, *similarly* $b\mathbf{C} \cap d\mathbf{C} \neq \emptyset \Leftrightarrow b\mathbf{C} = d\mathbf{C}$, *and* $\mathbf{C}b\mathbf{C} \cap \mathbf{C}d\mathbf{C} \neq \emptyset \Leftrightarrow \mathbf{C}b\mathbf{C} = \mathbf{C}d\mathbf{C}$. *In particular,*

$$\{\mathbf{C}b |\ b \in \mathbf{B}\},\quad \{b\mathbf{C} |\ b \in \mathbf{B}\},\quad \{\mathbf{C}b\mathbf{C} |\ b \in \mathbf{B}\}$$

*are partitions of* $\mathbf{B}$.

(ii) $(\mathbf{C}b\mathbf{C})^* = \mathbf{C}b^*\mathbf{C}$.

(iii) *If* $\mathbf{C}b = b\mathbf{C}$ *for each* $b \in \mathbf{B}$, *then* $\mathbf{C}b\mathbf{C} = \mathbf{C}b$.

*Proof:* Let $a \in \mathbf{C}b$ be an arbitrary element. Then $\lambda_{cba} \neq 0$ for some $c \in \mathbf{C}$. Hence, by Proposition 2.2, $\lambda_{c^*ab} \neq 0$, i.e., $b \in \mathbf{C}a$.

Thus if $a \in \mathbf{C}b \cap \mathbf{C}d$, then

$$\mathbf{C}d \subset \mathbf{C}\mathbf{C}a = \mathbf{C}a \subset \mathbf{C}\mathbf{C}b = \mathbf{C}b.$$

Similarly, $\mathbf{C}b \subset \mathbf{C}d$, implying $\mathbf{C}b = \mathbf{C}d$.

Let now $x \in \mathbf{C}b\mathbf{C} \cap \mathbf{C}d\mathbf{C}$. Then $x \in \mathrm{Supp}(c_1bc_2)$ and $x \in \mathrm{Supp}(c_3dc_4)$ for suitable $c_1, c_2, c_3, c_4 \in \mathbf{C}$. This implies

$$b \in \mathrm{Supp}({c_1}^* x {c_2}^*), \quad d \in \mathrm{Supp}({c_3}^* x {c_4}^*).$$

Therefore,

$$\mathbf{C}b\mathbf{C} \subset \mathbf{C}\,\mathrm{Supp}({c_1}^* x {c_2}^*)\mathbf{C} \subset \mathbf{C}x\mathbf{C}.$$

Analogously

$$\mathbf{C}d\mathbf{C} \subset \mathbf{C}x\mathbf{C}.$$

Combining this with evident inclusions $\mathbf{C}x\mathbf{C} \subset \mathbf{C}b\mathbf{C}$, $\mathbf{C}x\mathbf{C} \subset \mathbf{C}d\mathbf{C}$ we obtain $\mathbf{C}b\mathbf{C} = \mathbf{C}d\mathbf{C}$.

Parts (ii) and (iii) are trivial consequences of the first one. ∎

Proposition 4.8 and Theorem 4.9 generalize Theorem 2 of [7]. The proof of part (ii) of Proposition 4.8 is similar to that of Theorem 2 of [7].

Proposition 4.8: *Let* $\mathbf{C} \leq \mathbf{B}$ *be a closed subset and* $b \in \mathbf{B}$ *be an arbitrary element. Then:*

(i) $\forall_{c \in \mathbf{C}}\, c \cdot (\mathbf{C}b)^+ = |c|\,(\mathbf{C}b)^+$ *and* $(b\mathbf{C})^+ \cdot c = |c|\,(b\mathbf{C})^+$;

(ii) $\mathbf{C}^+ \cdot b = \alpha \cdot (\mathbf{C}b)^+$, $b \cdot \mathbf{C}^+ = \beta \cdot (b\mathbf{C})^+$ *for suitable* $\alpha, \beta \in \mathbb{R}$;

(iii) $\mathbf{C}^+ \cdot b \cdot \mathbf{C}^+ = \mu \cdot (\mathbf{C}b\mathbf{C})^+$ *for a suitable* $\mu \in \mathbb{R}$;

(iv) $|(\mathbf{C}b)^+| \geq |\mathbf{C}^+|$ $(|(b\mathbf{C})^+| \geq |\mathbf{C}^+|)$, *where equality holds if and only if* $\mathrm{Supp}(bb^*) \subset \mathbf{C}$ *(resp.* $\mathrm{Supp}(b^*b) \subset \mathbf{C}$*).*

*Proof:* (i) Since $c \in L_{\mathbf{C}b} \cap R_{b\mathbf{C}}$, (i) is an immediate consequence from Proposition 4.2.

(ii) Denote $\mathbf{C}b = \{b_1 = b, ..., b_m\}$ just for convenience. We have to show that $\mathbf{C}^+ \cdot b = \mu\,(\sum_{i=1}^m b_i)$. Since the vector space spanned by $b_i, 1 \leq i \leq m$ is invariant under left multiplication by the elements from $\mathbf{C}$, $\mathbf{C}^+ \cdot b_i = \sum_{j=1}^m \mu_{ij} b_j, \mu_{ij} \in \mathbb{R}$. Set $M := (\mu_{ij})_{1 \leq i,j \leq m}$. Since $(\mathbf{C}^+)^2 = |\mathbf{C}^+|\mathbf{C}^+$, $M^2 = |\mathbf{C}|M$. It follows from part (i) that $\sum_{i=1}^m \mu_{ij} = |\mathbf{C}|$ for each $1 \leq j \leq m$. The elements of matrix $M$

are non-negative real numbers. Moreover, $\mathbf{C}b = \{b_1, ..., b_m\}$ implies $\mu_{1j} \neq 0$ for all $1 \leq j \leq m$. Hence, $M$ is an indecomposable matrix with non-negative real entries. By the Perron–Frobenius Theorem, $|\mathbf{C}|$ is a maximal eigenvalue of $M$ and the $|\mathbf{C}|$-eigenspace is one-dimensional. The equality $M^2 = |\mathbf{C}|M$ implies that the columns of $M$ are $|\mathbf{C}|$-eigenvectors of $M$. Therefore, they are proportional, i.e., $\mathrm{rank}(M) = 1$. But the sum of the elements of every column is $|\mathbf{C}|$, whence all columns of $M$ are equal, i.e., $\mu_{i1} = \cdots = \mu_{im}$ for each $1 \leq i \leq m$.

(iii) $\mathbf{C}^+ \cdot b \cdot \mathbf{C}^+ = \sum_{x \in \mathbf{C}b\mathbf{C}} \mu_x x$. We have to show that all $\mu_x$ are equal to each other. By part (ii) of our claim

$$\mathbf{C}^+ \cdot b \cdot \mathbf{C}^+ = \alpha\,(\mathbf{C}b)^+\,\mathbf{C}^+ = \alpha \sum_{x \in \mathbf{C}b} \alpha_x (x\mathbf{C})^+, \quad \alpha, \alpha_x \in \mathbb{R}.$$

Therefore $\mu_x = \mu_y$ if $x$ and $y$ lie at the same right $\mathbf{C}$-coset. Analogously, $\mu_x = \mu_y$ if $x$ and $y$ lie at the same left $\mathbf{C}$-coset. Let now $x, y \in \mathbf{C}b\mathbf{C}$ be two arbitrary elements. Then $y \in \mathrm{Supp}(c_1 x c_2)$ for some $c_1, c_2 \in \mathbf{C}$. There exists $z \in \mathbf{B}$ such that $z \in \mathrm{Supp}(c_1 x)$ and $y \in \mathrm{Supp}(z c_2)$. Hence $\mu_x = \mu_y$, as desired.

(iv) It is enough to prove the claim only for right cosets. According to part (iii) of Proposition 2.3 the coefficient $\beta$ in (ii) is not greater than $|b|$. Hence $|(b\mathbf{C})^+| \geq |\mathbf{C}^+|$. If $\beta = |b|$, then, by Proposition 2.3, part (ii), we have $\mathrm{Supp}(b^* b) \subset \mathbf{C}$. ∎

*Remark:* The latter part of the Proposition implies that $|\mathbf{C}b\mathbf{C}| \geq |\mathbf{C}|$.

For each element $b \in \mathbf{B}$ we denote by $b/\!/\mathbf{C}$ the following expression:

$$b/\!/\mathbf{C} = |\mathbf{C}^+|^{-1} \cdot (\mathbf{C}b\mathbf{C})^+ = |\mathbf{C}^+|^{-1} \sum_{d \in \mathbf{C}b\mathbf{C}} d.$$

THEOREM 4.9: *Let $\mathbf{C} \leq \mathbf{B}$ be a closed subset and $b_1 = \mathbf{1}, b_2, \ldots, b_k$ be a complete set of representatives of $\mathbf{C}$-double cosets. Then the vector space spanned by the elements $b_i/\!/\mathbf{C}, 1 \leq i \leq k$ is a standard real GT-algebra with a distinguished basis $\mathbf{B}/\!/\mathbf{C} = \{b_i/\!/\mathbf{C} |\ 1 \leq i \leq k\}$. The structure constants of this GT-algebra are given by the following formula:*

$$\gamma_{ijk} = |\mathbf{C}^+|^{-1} \sum_{r \in \mathbf{C}b_i\mathbf{C}, s \in \mathbf{C}b_j\mathbf{C}} \lambda_{rst}, \tag{6}$$

*where $t \in \mathbf{C}b_k\mathbf{C}$ is an arbitrary element.*

*Proof:* Since $e_{\mathbf{C}} = |\mathbf{C}^+|^{-1}\mathbf{C}^+$ is an idempotent, the vector space $e_{\mathbf{C}} A e_{\mathbf{C}}$ spanned by the elements $e_{\mathbf{C}} \cdot b \cdot e_{\mathbf{C}}$, $b \in \mathbf{B}$ is a subalgebra of $A$ with the identity $e_{\mathbf{C}} = b_1/\!/\mathbf{C}$. By Proposition 4.8 (iii),

$$e_{\mathbf{C}} \cdot b \cdot e_{\mathbf{C}} = \mu |\mathbf{C}^+|^{-2}\,(\mathbf{C}b\mathbf{C})^+$$

for a suitable $\mu \in \mathbb{R}$. By Proposition 4.7 $(\mathbf{C}b\mathbf{C})^+ = (\mathbf{C}b_i\mathbf{C})^+$ for a suitable $1 \le i \le m$. Therefore, the elements $b_i/\!\!/\mathbf{C}$, $1 \le i \le k$ form a basis of the algebra $e_{\mathbf{C}}Ae_{\mathbf{C}}$.

If $\gamma_{ijk}$ are the structure constants of the algebra $e_{\mathbf{C}}Ae_{\mathbf{C}}$, then

$$b_i/\!\!/\mathbf{C} \cdot b_j/\!\!/\mathbf{C} = \sum_{k=1}^{m} \gamma_{ijk}\, b_k/\!\!/\mathbf{C},$$

or, equivalently,

$$(\mathbf{C}b_i\mathbf{C})^+ \cdot (\mathbf{C}b_j\mathbf{C})^+ = |\mathbf{C}^+| \sum_{k=1}^{m} \gamma_{ijk}\, (\mathbf{C}b_k\mathbf{C})^+ .$$

An element $t \in \mathbf{C}b_k\mathbf{C}$ appears in the left-hand side with the coefficient

$$\sum_{r \in \mathbf{C}b_i\mathbf{C}, s \in \mathbf{C}b_j\mathbf{C}} \lambda_{rst}.$$

Therefore,

$$|\mathbf{C}^+|\gamma_{ijk} = \sum_{r \in \mathbf{C}b_i\mathbf{C}, s \in \mathbf{C}b_j\mathbf{C}} \lambda_{rst}.$$

Thus an algebra $A/\!\!/\mathbf{C}$ with a distinguished basis $\mathbf{B}/\!\!/\mathbf{C}$ satisfies the axioms T0,T1. Since $e_{\mathbf{C}}{}^* = e_{\mathbf{C}}$, $A/\!\!/\mathbf{C}$ is $^*$-invariant. The equality $(\mathbf{C}b\mathbf{C})^* = (\mathbf{C}b^*\mathbf{C})$ shows that $(\mathbf{B}/\!\!/\mathbf{C})^* = \mathbf{B}/\!\!/\mathbf{C}$. Hence $A/\!\!/\mathbf{C}$ satisfies also T2.

In order to check T3 we compute $\gamma_{ij1}$. If $(\mathbf{C}b_i\mathbf{C})^* \neq \mathbf{C}b_j\mathbf{C}$, then $(\mathbf{C}b_i\mathbf{C})^* \cap \mathbf{C}b_j\mathbf{C} = \emptyset$ and (6) implies $\gamma_{ij1} = 0$. If $(\mathbf{C}b_i\mathbf{C})^* = \mathbf{C}b_j\mathbf{C}$, then

$$\gamma_{ij1} = |\mathbf{C}^+|^{-1} \sum_{x \in \mathbf{C}b_i\mathbf{C}} |x| = |\mathbf{C}^+|^{-1} \sum_{x \in \mathbf{C}b_i{}^*\mathbf{C}} |x| = \gamma_{ji1}$$

as desired.

Thus we have shown that $(A/\!\!/\mathbf{C}, \mathbf{B}/\!\!/\mathbf{C})$ is a real GT-algebra. To finish the proof we need to check that it is standard.

As we have seen before,

$$\gamma_{ii^*1} = |\mathbf{C}^+|^{-1} \sum_{x \in \mathbf{C}b_i\mathbf{C}} |x| = \frac{|(\mathbf{C}b\mathbf{C})^+|}{|\mathbf{C}^+|} = |b_i/\!\!/\mathbf{C}|.$$

But the map $b_i/\!\!/\mathbf{C} \mapsto |b_i/\!\!/\mathbf{C}|$ is the restriction of the degree homomorphism of $A$ onto $(A/\!\!/\mathbf{C}, \mathbf{B}/\!\!/\mathbf{C})$, implying that $b_i/\!\!/\mathbf{C} \mapsto \gamma_{ii^*1}$ is a homomorphism. ∎

*Remark:* If $A$ is a group algebra of a finite group $\mathbf{B}$, then $(A, \mathbf{B})$ is a real non-singular standard GT-algebra. $\mathbf{C} \subset \mathbf{B}$ is a closed subset if and only if $\mathbf{C}$ is a subgroup of $\mathbf{B}$. In this case $(A/\!/\mathbf{C}, \mathbf{B}/\!/\mathbf{C})$ is a Hecke algebra $H(\mathbf{C}, \mathbf{B})$.

The algebra $(A/\!/\mathbf{C}, \mathbf{B}/\!/\mathbf{C})$ will be called **the quotient** of $(A, \mathbf{B})$ by a closed subset $\mathbf{C}$. It follows from the proof that $o(\mathbf{C})o(\mathbf{B}/\!/\mathbf{C}) = o(\mathbf{B})$. If $(A, \mathbf{B})$ is a Bose–Mesner algebra of a homogeneous coherent configuration, then both $o(\mathbf{C})$ and $o(\mathbf{B}/\!/\mathbf{C})$ are rational integers [22], [20]. If $(A, \mathbf{B})$ is an arbitrary integral standard GT-algebra, then $o(\mathbf{B}/\!/\mathbf{C})$ may be not integral.

If $(A, \mathbf{B}), (C, \mathbf{D})$ are two GT-algebras with the same ring of scalars $R$, then one can define their **tensor product** $(A \otimes_R C, \mathbf{B} \otimes \mathbf{D})$ where the distinguished basis $\mathbf{B} \otimes \mathbf{D}$ is the set of tensors $b \otimes d$, $b \in \mathbf{B}$, $d \in \mathbf{D}$. It is easy to see that $(A \otimes_R C, \mathbf{B} \otimes \mathbf{D})$ satisfies all the axioms T0–T3. Moreover, if both $(A, \mathbf{B})$ and $(C, \mathbf{D})$ are standard then so is their tensor product. The same is true for real algebras.

If $(C, \mathbf{D})$ is standard, then $(A \otimes_R C, \mathbf{B} \otimes \mathbf{D})$ contains a subalgebra $(A, \mathbf{B}) \wr (C, \mathbf{D})$ spanned, as an $R$-module, by the following basis:

$$\mathbf{B} \wr \mathbf{D} = \{\mathbf{1} \otimes d \,|\, d \in \mathbf{D}\} \cup \{b \otimes \mathbf{D}^+ \,|\, b \in \mathbf{B}, b \neq \mathbf{1}\}.$$

A direct check shows that an $R$-submodule spanned by the above basis is a subalgebra of $(A \otimes_R C, \mathbf{B} \otimes \mathbf{D})$ that satisfies all the axioms. In what follows we shall denote it by $(A, \mathbf{B}) \wr (C, \mathbf{D})$ and call it **the wreath product** of $(C, \mathbf{D})$ by $(A, \mathbf{B})$. The dimension of $(A \wr C, \mathbf{B} \wr \mathbf{D})$ is always equal to $\dim(A) + \dim(C) - 1$.

Both constructions described above are well known in the theory of homogeneous coherent configurations and table algebras [4], [6], [20].

PROPOSITION 4.10: *Let $(A \wr C, \mathbf{B} \wr \mathbf{D})$ be the wreath product of two standard real algebras. Then:*

(i) $\overline{\mathbf{D}} = \{\mathbf{1} \otimes d \,|, d \in \mathbf{D}\} \leq \mathbf{B} \wr \mathbf{D}$ *and the GT-algebra spanned by* $\{\mathbf{1} \otimes d \,|, d \in \mathbf{D}\}$ *is isomorphic to* $(C, \mathbf{D})$.

(ii) *For each* $b \in \mathbf{B}$, $\overline{\mathbf{D}} \subset S_{b \otimes \mathbf{D}^+}$.

(iii) *The quotient algebra*

$$\left((A \wr C)/\!/\overline{\mathbf{D}}; (\mathbf{B} \wr \mathbf{D})/\!/\overline{\mathbf{D}}\right)$$

*is isomorphic to* $(A, \mathbf{B})$.

*Proof:* Part (i) follows directly from the definition of the wreath product, since $(\mathbf{1} \otimes d) \cdot (\mathbf{1} \otimes d') = \mathbf{1} \otimes dd'$ for every pair $d, d' \in \mathbf{D}$.

(ii) For each $d \in \mathbf{D}$, $(\mathbf{1} \otimes d) \cdot (b \otimes \mathbf{D}^+) = (b \otimes \mathbf{D}^+) \cdot (\mathbf{1} \otimes d) = |d|(b \otimes \mathbf{D}^+)$, hereby proving this part.

(iii) Since $x\overline{\mathbf{D}} = \overline{\mathbf{D}}x$ for each $x \in \mathbf{B} \wr \mathbf{D}$,

$$\overline{\mathbf{D}}x\overline{\mathbf{D}} = x\,\overline{\mathbf{D}} = \begin{cases} \overline{\mathbf{D}} & \text{if } x = \mathbf{1} \otimes d, \quad d \in \mathbf{D}, \\ \{x\} & \text{if } x = b \otimes \mathbf{D}^+, \quad b \neq \mathbf{1}. \end{cases}$$

Thus we have $\dim(A)$ double cosets, each of them of the form $b \otimes \mathbf{D}$. Now it is easy to check that the mapping

$$\frac{1}{|\mathbf{D}^+|}(b \otimes \mathbf{D})^+ \mapsto b$$

is an isomorphism between GT-algebras

$$\left((A \wr C)/\!\!/\overline{\mathbf{D}}; (\mathbf{B} \wr \mathbf{D})/\!\!/\overline{\mathbf{D}}\right)$$

and $(A, \mathbf{B})$. ∎

## 5. Integral standard GT-algebras

A non-singular real GT-algebra is called **integral** if its structure constants $\lambda_{abc}$ and the degrees $f(b)$ are integers. An integral commutative GT-algebra is merely a classical integral table algebra [9].

In this section we always assume that $(A, \mathbf{B})$ is an integral standard GT-algebra. If $d = \dim(A)$, then we can number the elements of $\mathbf{B}$ in such a way that the sequence of **the degrees** $|b_1| = 1, |b_2|, ..., |b_d|$ is not decreasing, i.e., $|b_i| \leq |b_{i+1}|$.

We start from the following simple claim:

PROPOSITION 5.1: *Let $(A, \mathbf{B})$ be an integral non-singular standard GT-algebra. Then:*

(i) $\lambda_{abc}|c| \equiv 0 \,(\mathrm{mod}\ \ \mathrm{lcm}(|a||b|))$;

(ii) $|\,\mathrm{Supp}(ab)| \leq \gcd(|a|, |b|)$, *in particular*, $|\,\mathrm{Supp}(ab)| = 1$ *for relatively prime* $|a|$ *and* $|b|$.

*Proof:* (i) By Proposition 2.2

$$\lambda_{abc}|c| = \lambda_{c^*ab^*}|b| = \lambda_{cb^*a}|a|,$$

and the claim follows.

(ii) For each $c \in \mathrm{Supp}(ab)$, $\lambda_{abc}|c| \neq 0$. Together with the previous part of the claim this implies $\lambda_{abc}|c| \geq \mathrm{lcm}(|a|,|b|)$ for every $c \in \mathrm{Supp}(ab)$. Therefore,

$$|\,\mathrm{Supp}(ab)| \leq \frac{|a||b|}{\mathrm{lcm}(|a|,|b|)} = \gcd(|a|,|b|). \qquad \blacksquare$$

PROPOSITION 5.2: *Let $a \in \mathbf{B}$ be a faithful element of an integral standard GT-algebra. Then*

$$\forall_p \left(p \in \pi(\mathbf{B}) \Rightarrow p \leq |a|\right).$$

*Proof:* Assume the contrary, i.e., there exists a prime $p \in \pi(\mathbf{B})$ such that $p > |a|$. Define $\mathbf{B}_0 = \{b \in \mathbf{B} \mid p \not| \, |b|\}$, $\mathbf{B}_1 = \{b \in \mathbf{B} \mid p| \, |b|\}$. Clearly, $\mathbf{B}$ is a disjoint union of $\mathbf{B}_0$ and $\mathbf{B}_1$. Since $a \in \mathbf{B}_0$ and $p \in \pi(\mathbf{B})$, the above sets are non-empty. By Proposition 4.3 there exist $b \in \mathbf{B}_0$ and $c \in \mathbf{B}_1$ such that $\lambda_{acb} \neq 0$. By Proposition 5.1

$$\lambda_{acb}|b| \equiv 0(\mathrm{mod}\ \mathrm{lcm}(|a|,|c|)) \Rightarrow \lambda_{acb}|b| \equiv 0(\mathrm{mod}\ p|a|).$$

On the other hand, $\lambda_{acb} \leq |a| < p$. Therefore $|b| \equiv 0(\mathrm{mod}\ p)$, yielding a contradiction. $\blacksquare$

*Remark:* For homogeneous coherent configurations this fact was proven in [18]. The proof given in [18] was reproduced in [12]. For integral standard table algebras it was proven by H. Blau in [8]. Here we gave the proof that appeared in [18].

Let $a \in \mathbf{B}$ be an arbitrary element. We define $a^{[0]} = \mathbf{1}$ and, for each natural $l \in \mathbb{N}$, we set

$$a^{[l]} = \begin{cases} a^{[l-1]}a^* & \text{if } l \text{ is even,} \\ a^{[l-1]}a & \text{otherwise.} \end{cases}$$

Now we can define two functions on $\mathbf{B}$:

$$d_a(b) = \min\{n \mid b \in \mathrm{Supp}(a^n)\}; \quad \partial_a(b) = \min\{n \mid b \in \mathrm{Supp}(a^{[n]})\}.$$

It is clear that $a = a^*$ implies $\partial_a = d_a$.

In the case, when $\forall_{n\in\mathbb{Z}} (b \notin \mathrm{Supp}(a^n))$ (respectively $\forall_{n\in\mathbb{Z}} b \notin \mathrm{Supp}(a^{[n]})$) we shall write $d_a(b) = \infty$ (resp. $\partial_a(b) = \infty$). We shall say that an element $a \in \mathbf{B}$ is **strongly faithful** if $\partial_a(x) \neq \infty$ for all $x \in \mathbf{B}$. It is easy to see that each strongly faithful element is also faithful.

PROPOSITION 5.3: *An element $a \in \mathbf{B}$ is strongly faithful if it satisfies at least one of the following conditions:*
(i) *$a$ is faithful and $a = a^*$;*
(ii) *the closed subset generated by $\mathrm{Supp}(aa^*)$ coincides with $\mathbf{B}$.*

*Proof:* Straightforward.

PROPOSITION 5.4: *If $\partial_a(b) = l \geq 2$, then there exists $c \in \mathbf{B}$ such that:*
(i) $\partial_a(c) \leq l-1$,
(ii) *$b \in \mathrm{Supp}(c\hat{a})$ and $|\mathrm{Supp}(c\hat{a})| \geq 2$,*
*where $\hat{a} = a^*$ if $l$ is even and $\hat{a} = a$ otherwise.*

*Proof:* By definition, $b \in \mathrm{Supp}(a^{[l]}) = \mathrm{Supp}(a^{[l-1]})\hat{a}$. Therefore, there exists $c \in \mathrm{Supp}(a^{[l-1]})$ such that $b \in \mathrm{Supp}(c\hat{a})$. Clearly, $\partial_a(c) \leq l-1$.

Since $c \in \mathrm{Supp}(a^{[l-1]})$, there exists $d \in \mathrm{Supp}(a^{[l-2]})$ such that $c \in \mathrm{Supp}(d\hat{a}^*)$, or, equivalently, $d \in \mathrm{Supp}(c\hat{a})$. The elements $d$ and $b$ are different, because of $\partial_a(b) = l, \partial_a(d) \leq l-2$. Therefore, $\{b, d\} \subset \mathrm{Supp}(c\hat{a})$, as desired. ∎

It is easy to see that $\partial_a(c)$ exists for each $c \in \mathbf{B}$ if $\mathrm{Supp}(aa^*)$ generates $\mathbf{B}$.

The claim below was first proved by H. Blau in [8] in the case of $a = a^*$.

PROPOSITION 5.5: *Assume that $b_2$ is strongly faithful. If $|b_2| = p$, where $p$ is prime, then $p^2 \not| |b_j|$ for all $1 \leq j \leq d$.*

*Proof:* Assume that $\mathbf{C} = \{c \in \mathbf{B} \mid p^2 \mid |c|\} \neq \emptyset$. Take $c \in \mathbf{C}$ with the minimal value $l := \partial_{b_2}(c)$. Clearly $l \geq 2$. By Proposition 5.4 there exists $f \in \mathbf{B}$ such that $\partial_{b_2}(f) \leq l-1, |\mathrm{Supp}(f\hat{b_2})| \geq 2$ and $c \in \mathrm{Supp}(f\hat{b_2})$ (here $\hat{}$ has the same meaning as in the previous statement). By Proposition 5.1 $\lambda_{f\hat{b}_2c}|c| \equiv 0(\mathrm{mod}\ \ \mathrm{lcm}(|b_2|, |f|))$. Combining this with $|c| \equiv 0(\mathrm{mod}\ p^2)$ we obtain

$$\lambda_{f\hat{b}_2c}|c| \equiv 0(\mathrm{mod}\ \ \mathrm{lcm}(|f|, p^2)),$$

implying $\lambda_{f\hat{b}_2c}|c| \geq \ \mathrm{lcm}(p^2, |f|)$.

Since $\partial_{b_2}(f) \leq l-1$, $f \notin \mathbf{C}$, and therefore $p^2 \not| |f|$, whence $\mathrm{lcm}(|f|, p^2) \geq p|f|$. Thus $\lambda_{f\hat{b}_2c}|c| \geq p|f|$.

On the other hand, $\lambda_{f\hat{b}_2c}|c| \leq |b_2||f| = p|f|$. Therefore, $b_2 f = \lambda_{f\hat{b}_2c} c$ and $\mathrm{Supp}(b_2 f) = \{c\}$. This is a contradiction. ∎

PROPOSITION 5.6: *If $b_2$ is a strongly faithful element of $\mathbf{B}$, then*

$$\frac{|b_i|}{|b_{i-1}|} \leq |b_2| - 1$$

*holds for each* $3 \leq i \leq d$.

*Proof:* Among all $b \in \mathbf{B}$ that satisfy the inequality $|b| > |b_{i-1}|$ we choose an element $c$ with a minimal value of $\partial_{b_2}(c)$. By Proposition 5.4 there exists $f \in \mathbf{B}$ such that $\partial_{b_2}(f) < \partial_{b_2}(c), c \in \mathrm{Supp}(f\hat{b_2})$ and there is some $c_1 \neq c$ in $\mathrm{Supp}(f\hat{b_2})$. Clearly, $|f| \leq |b_{i-1}|$. So

$$|f||\hat{b_2}| \geq \lambda_{f\hat{b_2}c}|c| + \lambda_{f\hat{b_2}c_1}|c_1| \geq |c| + |f|,$$

as $\lambda_{f\hat{b_2}c_1}|c_1| \equiv 0(\mathrm{mod}\ |f|)$. It follows that $|c| \leq (|b_2| - 1)|f| \leq (|b_2| - 1)|b_{i-1}|$. ∎

PROPOSITION 5.7: *Let $(A, \mathbf{B})$ be an integral standard GT-algebra. Assume that there exist $b_i, b_j \in \mathbf{B}$ such that $|b_i| > 1, \gcd(|b_i|, |b_j|) = 1$. By Proposition 5.1,* $\mathrm{Supp}(b_i b_j) = \{b_k\}$ *for a suitable $b_k \in \mathbf{B}$. Assume that $|b_k| = |b_j|$. Then:*

(i) $\mathrm{Supp}(b_i{}^* b_i) \subset L_{b_j}$.

(ii) *If $b_i b_i{}^* b_i = b_i{}^* b_i^2$, then $\mathbf{B}_{b_i} \neq \mathbf{B}$.*

(iii) *If $|b_j| = |b_d|$, then $\mathbf{B}_{b_i} \neq \mathbf{B}$.*

*Proof:* (i) Follows directly from Proposition 4.4.

(ii) Set $\mathbf{V} := \{c \in \mathbf{B} |\ b_i c = |b_i| f$ for some $f \in \mathbf{B}$ and $|c| = |b_j|\}$. Clearly $\mathbf{V} \neq \emptyset$.

We claim that $\mathrm{Supp}(b_i\mathbf{V}) = \mathbf{V}$. Indeed, if $f \in \mathrm{Supp}(b_i\mathbf{V})$, then there exists $c \in \mathbf{V}$ such that $b_i c = |b_i| f$. Since $|f| = |c| = |b_j|$, we obtain that $\gcd(|b_i|, |f|) = 1$, from which it follows that $b_i f = \mu f'$, $\mu \leq |b_i|$, $f' \in \mathbf{B}$. We have

$$b_i{}^* b_i f = \frac{1}{|b_i|} b_i{}^* b_i b_i c = \frac{1}{|b_i|} b_i b_i{}^* b_i c = |b_i| b_i c = |b_i|^2 f.$$

Now Proposition 4.4 implies $\mu = |b_i|$.

On the other hand, $b_i{}^* b_i f = \mu b_i{}^* f'$, implying $b_i{}^* f' = |b_i|^2/\mu f$. Hence $|b_i|^2/\mu \leq |b_i|$ and, consequently, $\mu = |b_i|$.

Thus $f \in \mathbf{V}$ and $\mathrm{Supp}(b_i\mathbf{V}) = \mathbf{V}$. This immediately implies $b_i \in L_{\mathbf{V}}$. Since $\mathbf{V} \neq \mathbf{B}$, $L_{\mathbf{V}} \neq \mathbf{B}$, implying part (ii) of our claim.

(iii) We set $\mathbf{V} := \{c \in \mathbf{B} |\ |c| = |b_j|\}$. As before, it is enough to show that $\mathrm{Supp}(b_i\mathbf{V}) = \mathbf{V}$. Take an arbitrary $c \in \mathbf{V}$. Then $\gcd(|c|, |b_i|) = 1$ implies that $b_i c = \mu f$ for some $f \in \mathbf{B}$. Taking into account that $|f| \leq |b_d| = |b_j| = |c|$ and $\mu \leq |b_i|$ we obtain $|f| = |b_d|$. Thus $\mathrm{Supp}(b_i\mathbf{V}) = \mathbf{V}$, as desired. ∎

*Remark:* For related results on finite groups see [17].

PROPOSITION 5.8: *Let $(A, \mathbf{B})$ be a standard integral GT-algebra. Assume that $\mathbf{B}$ contains a faithful element $b \in \mathbf{B}$ with $|b| = 2$ and $|x| \geq 2$ for all $x \in \mathbf{B}$, $x \neq 1$. Then there exists a natural $n$ such that $A \subset \mathbb{Z}C_{2n+1}$ with a distinguished basis $\mathbf{B} = \{g^i + g^{-i} \,|\, i = 1, \ldots, n\}$ where $g$ is a generator of $C_{2n+1}$.*

*Proof:* Let $b_1 = 1, b_2 = b, \ldots, b_m$ be the ordering of the elements of $\mathbf{B}$ according to their degrees (i.e., $|b_i| \leq |b_{i+1}|$). At first we show that $\mathrm{Supp}(bb^*)$ generates $\mathbf{B}$.

Assume that it is not true. Then $bb^* = 2 \cdot \mathbf{1} + c$ for some $c \in \mathbf{B}$, $|c| = 2$, $c = c^*$ and $|\mathbf{B}_c| \neq |\mathbf{B}|$. By induction on $|\mathbf{B}|$, $\mathbf{B}_c$ satisfies the conclusion of Proposition 5.8. Therefore, there exists some $d \in \mathbf{B}_c$ with $bb^* = 2 \cdot \mathbf{1} + c = dd^*$. Then $(d^*b, d^*b) = (bb^*, dd^*) = 6$. Since $b \neq d$, either $d^*b \in \mathbf{B}$ or $d^*b = u + v$ for suitable $u, v \in \mathbf{B}, |u| = |v| = 2$ ($u, v$ may be equal). Therefore $(d^*b, d^*b) \in \{4, 8\}$, a contradiction.

Thus we may assume that $\mathrm{Supp}(bb^*)$ generates $\mathbf{B}$. By Proposition 5.6, $|b_i| \leq |b_{i-1}|$. Therefore $|b_i| = 2$ for each $i = 2, \ldots, m$.

Since all non-trivial elements of $\mathbf{B}$ are of degree 2, for each $x \in \mathbf{B}^{\#}$ there exists a unique $f(x) \in \mathbf{B}^{\#}$ with the property $xx^* = 2 \cdot 1 + f(x)$. We claim that $f$ is injective. Indeed, if $f(x) = f(y)$, then $(xx^*, yy^*) = 6$, implying a contradiction. Thus, $f$ is injective and therefore surjective (we remind the reader that $\mathbf{B}$ is finite). Since $f(x)^* = f(x)$, each element $x \in \mathbf{B}$ is real, whence $A$ is commutative.

All integral table algebras with faithful element of degree 2 were classified in [9] provided that either the given element is real or there is no non-identity linear element. Using this classification one can easily complete the proof. ∎

*Proof of Theorem 1.3:* (i) If $|b_2| = 1$, then $b_2$ is linear and therefore $\mathsf{L}(\mathbf{B}) \neq \{\mathbf{1}\}$. Since $\mathsf{L}(\mathbf{B}) \leq \mathbf{B}$, $\mathsf{L}(\mathbf{B}) = \mathbf{B}$. Thus $A$ is a group algebra of the group $\mathbf{B}$. By primitivity of $A$, $\mathbf{B}$ has no subgroup. Therefore, $\mathbf{B}$ is of prime order.

(ii) is a direct consequence of Proposition 5.2.

(iii) Assume that $\gcd(|b_i|, |b_j|) = 1$ for some $1 < i, j$. Then, by Proposition 5.1 part (i), $|\mathrm{Supp}(b_i b_j)| = 1$, whence $b_i b_j = \mu b_k, \mu \in \mathbb{Z}$ for a suitable $b_k \in \mathbf{B}$. WLOG we may assume that $|b_i| \leq |b_j|$. If $|b_j| < |b_k|$, then we are done.

Assume now that $|b_j| = |b_k|$. Since $|b_i| \geq |b_2|$, $|b_i| > 1$, implying that $\mathrm{Supp}(b_i b_i{}^*) \neq \{\mathbf{1}\}$.

$\mathbf{B}$ is primitive, hence $\mathsf{L}(\mathbf{B}) = \{\mathbf{1}\} = L_{b_j}$. But by Proposition 5.7(i), $\{\mathbf{1}\} \subsetneq \mathrm{Supp}(b_i{}^* b_i) \subset L_{b_j}$, which is a contradiction.

(iv) follows from Proposition 5.5.

(v) follows from Proposition 5.6.

(vi) follows from part (iii) of this claim.

(vii) follows directly from Proposition 5.8. ∎

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